ON THE SPHERE AND CYLINDER.

BOOK II.

"ARCHIMEDES to Dositheus greeting.

On a former occasion you asked me to write out the proofs of the problems the enunciations of which I had myself sent to Conon. In point of fact they depend for the most part on the theorems of which I have already sent you the demonstrations, namely (1) that the surface of any sphere is four times the greatest circle in the sphere, (2) that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of its base, (3) that the cylinder whose base is the greatest circle in any sphere and whose height is equal to the diameter of the sphere is itself in magnitude half as large again as the sphere, while its surface [including the two bases] is half as large again as the surface of the sphere, and (4) that any solid sector is equal to a cone whose base is the circle which is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere. Such then of the theorems and problems as depend on these theorems I have written out in the book which I send herewith; those which are discovered by means of a different sort of investigation, those namely which relate to spirals and the conoids, I will endeavour to send you soon.

The first of the problems was as follows: *Given a sphere, to find a plane area equal to the surface of the sphere.*

The solution of this is obvious from the theorems aforesaid. For four times the greatest circle in the sphere is both a plane area and equal to the surface of the sphere.

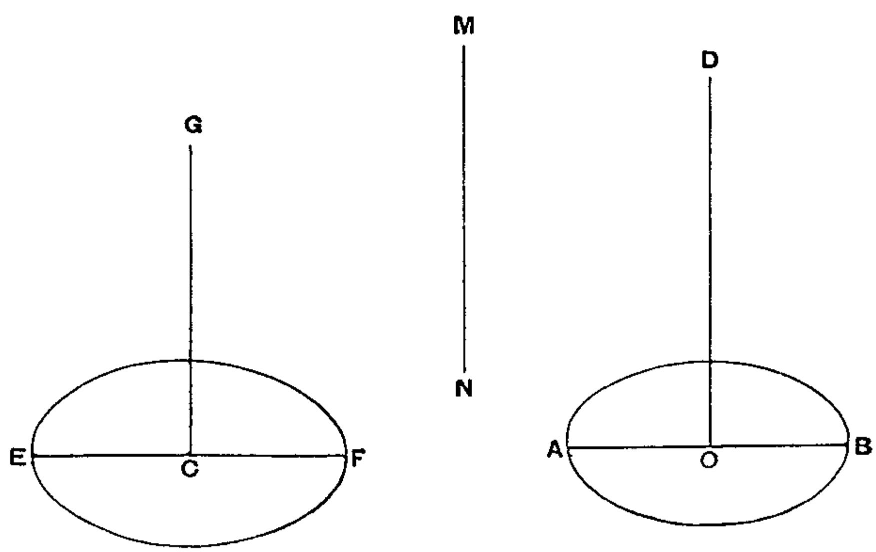
The second problem was the following."

Proposition 1. (Problem.)

Given a cone or a cylinder, to find a sphere equal to the cone or to the cylinder.

If V be the given cone or cylinder, we can make a cylinder equal to $\frac{3}{2}V$. Let this cylinder be the cylinder whose base is the circle on AB as diameter and whose height is OD.

Now, if we could make another cylinder, equal to the cylinder (OD) but such that its height is equal to the diameter of its base, the problem would be solved, because this latter cylinder would be equal to $\frac{3}{2}V$, and the sphere whose diameter is equal to the height (or to the diameter of the base) of the same cylinder would then be the sphere required [I. 34, Cor.].



Suppose the problem solved, and let the cylinder (CG) be equal to the cylinder (OD), while EF, the diameter of the base, is equal to the height CG.

Then, since in equal cylinders the heights and bases are reciprocally proportional,

$$AB^2 : EF^2 = CG : OD$$
$$= EF : OD \quad\quad (1).$$

Suppose MN to be such a line that

$$EF^2 = AB \cdot MN \quad\quad (2).$$

Hence $$AB : EF = EF : MN,$$

and, combining (1) and (2), we have

$$AB : MN = EF : OD,$$

or $$AB : EF = MN : OD.$$

Therefore $$AB : EF = EF : MN = MN : OD,$$

and EF, MN *are two mean proportionals between* AB, OD.

The synthesis of the problem is therefore as follows. Take two mean proportionals EF, MN between AB and OD, and describe a cylinder whose base is a circle on EF as diameter and whose height CG is equal to EF.

Then, since

$$AB : EF = EF : MN = MN : OD,$$
$$EF^2 = AB \cdot MN,$$

and therefore $$AB^2 : EF^2 = AB : MN$$
$$= EF : OD$$
$$= CG : OD;$$

whence the bases of the two cylinders (OD), (CG) are reciprocally proportional to their heights.

Therefore the cylinders are equal, and it follows that

$$\text{cylinder } (CG) = \tfrac{3}{2} V.$$

The sphere on EF as diameter is therefore the sphere required, being equal to V.

Proposition 2.

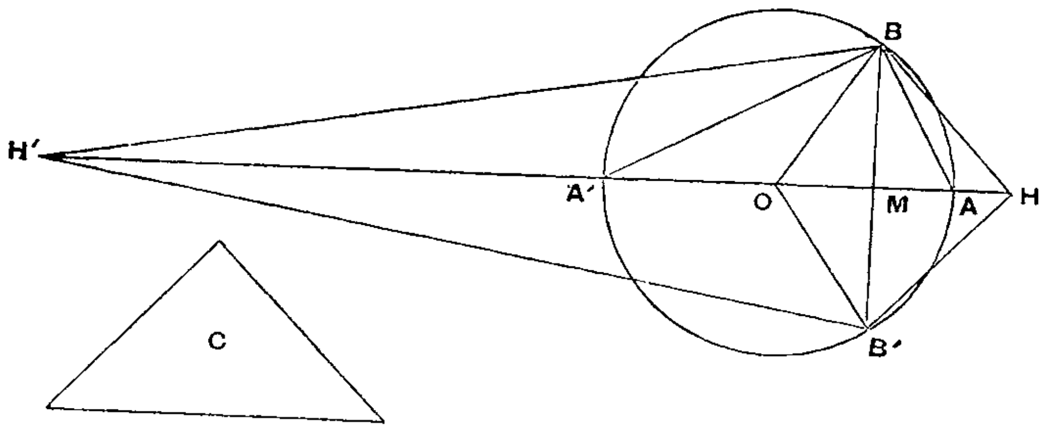
If BAB′ be a segment of a sphere, BB′ a diameter of the base of the segment, and O the centre of the sphere, and if AA′ be the diameter of the sphere bisecting BB′ in M, then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is h, where

$$h : AM = OA' + A'M : A'M.$$

Measure MH along MA equal to h, and MH' along MA' equal to h', where

$$h' : A'M = OA + AM : AM.$$

Suppose the three cones constructed which have O, H H' for their apices and the base (BB') of the segment for their common base. Join AB, $A'B$.



Let C be a cone whose base is equal to the surface of the segment BAB' of the sphere, i.e. to a circle with radius equal to AB [I. 42], and whose height is equal to OA.

Then the cone C is equal to the solid sector $OBAB'$ [I. 44].

Now, since $HM : MA = OA' + A'M : A'M$,

dividendo, $HA : AM = OA : A'M$,

and, alternately, $HA : AO = AM : MA'$,

so that

$$HO : OA = AA' : A'M$$
$$= AB^2 : BM^2$$
$$= (\text{base of cone } C) : (\text{circle on } BB' \text{ as diameter}).$$

But OA is equal to the height of the cone C; therefore, since cones are equal if their bases and heights are reciprocally proportional, it follows that the cone C (or the solid sector $OBAB'$) is equal to a cone whose base is the circle on BB' as diameter and whose height is equal to OH.

And this latter cone is equal to the sum of two others having the same base and with heights OM, MH, i.e. to the solid rhombus $OBHB'$.

Hence the sector $OBAB'$ is equal to the rhombus $OBHB'$.

Taking away the common part, the cone OBB',

$$\text{the segment } BAB' = \text{the cone } HBB'.$$

Similarly, by the same method, we can prove that

$$\text{the segment } BA'B' = \text{the cone } H'BB'.$$

Alternative proof of the latter property.

Suppose D to be a cone whose base is equal to the surface of the whole sphere and whose height is equal to OA.

Thus D is equal to the volume of the sphere. [I. 33, 34]

Now, since $OA' + A'M : A'M = HM : MA$,

dividendo and *alternando*, as before,

$$OA : AH = A'M : MA.$$

Again, since $H'M : MA' = OA + AM : AM$,

$$H'A' : OA = A'M : MA$$
$$= OA : AH, \text{ from above.}$$

Componendo, $H'O : OA = OH : HA$ ....(1).

Alternately, $H'O : OH = OA : AH$ ....(2),

and, *componendo*, $HH' : HO = OH : HA$,

$$= H'O : OA, \text{ from (1),}$$

whence $HH' . OA = H'O . OH$..................(3).

Next, since $H'O : OH = OA : AH$, by (2),

$$= A'M : MA,$$

$$(H'O + OH)^2 : H'O . OH = (A'M + MA)^2 : A'M . MA,$$

whence, by means of (3),

$$HH'^2 : HH' . OA = AA'^2 : A'M . MA,$$

or

$$HH' : OA = AA'^2 : BM^2.$$

Now the cone D, which is equal to the sphere, has for its base a circle whose radius is equal to AA', and for its height a line equal to OA.

Hence this cone D is equal to a cone whose base is the circle on BB' as diameter and whose height is equal to HH';

therefore the cone $D =$ the rhombus $HBH'B'$,

or the rhombus $HBH'B' =$ the sphere.

But the segment $BAB' =$ the cone HBB';

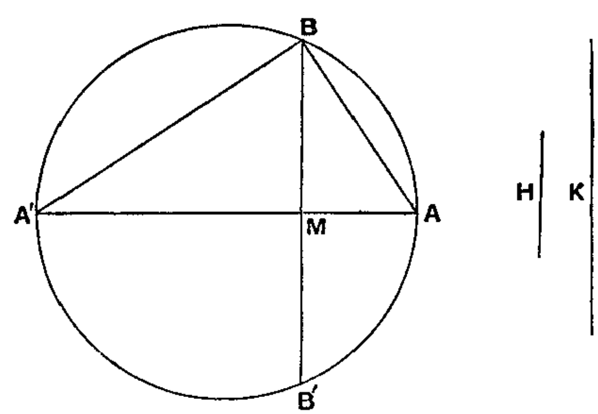
therefore the remaining segment $BA'B' =$ the cone $H'BB'$.

Cor. *The segment BAB' is to a cone with the same base and equal height in the ratio of $OA' + A'M$ to $A'M$.*

Proposition 3. (Problem.)

To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio.

Suppose the problem solved. Let AA' be a diameter of a great circle of the sphere, and suppose that a plane perpendicular to AA' cuts the plane of the great circle in the straight



line BB', and AA' in M, and that it divides the sphere so that the surface of the segment BAB' has to the surface of the segment $BA'B'$ the given ratio.

Now these surfaces are respectively equal to circles with radii equal to AB, $A'B$ [I. 42, 43].

Hence the ratio $AB^2 : A'B^2$ is equal to the given ratio, i.e. AM is to MA' in the given ratio.

Accordingly the synthesis proceeds as follows.

If $H : K$ be the given ratio, divide AA' in M so that

$$AM : MA' = H : K.$$

Then $AM : MA' = AB^2 : A'B^2$

$= (\text{circle with radius } AB) : (\text{circle with radius } A'B)$

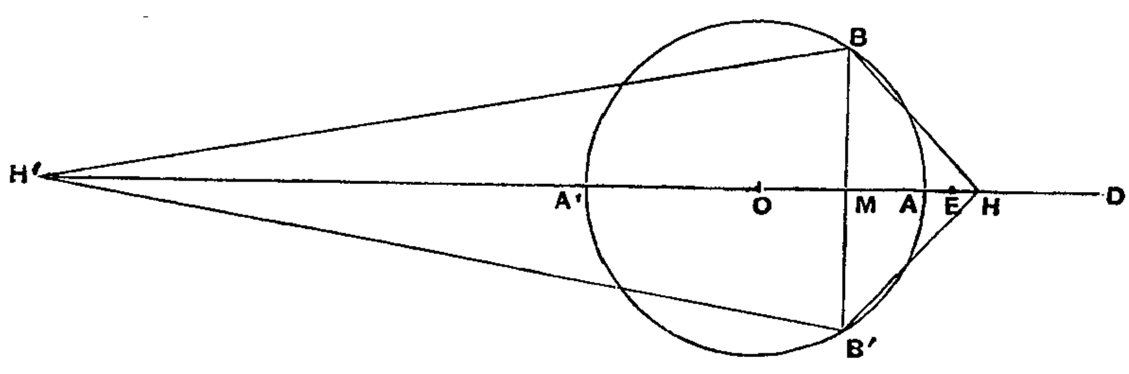
$= (\text{surface of segment } BAB') : (\text{surface of segment } BA'B')$.

Thus the ratio of the surfaces of the segments is equal to the ratio $H : K$.

Proposition 4. (Problem.)

To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio.

Suppose the problem solved, and let the required plane cut the great circle ABA' at right angles in the line BB'. Let AA' be that diameter of the great circle which bisects BB' at right angles (in M), and let O be the centre of the sphere.



Take H on OA produced, and H' on OA' produced, such that

$$OA' + A'M : A'M = HM : MA, \qquad \text{..........(1)},$$

and

$$OA + AM : AM = H'M : MA' \qquad \text{..........(2)}.$$

Join BH, $B'H$, BH', $B'H'$.

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