QUADRATURE OF THE PARABOLA.

"ARCHIMEDES to Dositheus greeting.

"When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone* and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration

* There appears to be some corruption here: the expression in the text is *τᾶς ὅλου τοῦ κώνου τομᾶς*, and it is not easy to give a natural and intelligible meaning to it. The section of 'the whole cone' might perhaps mean a section cutting right through it, i.e. an ellipse, and the 'straight line' might be an axis or a diameter. But Heiberg objects to the suggestion to read *τᾶς ὀξυγωνίου κώνου τομᾶς*, in view of the addition of *καὶ εὐθείας*, on the ground that the former expression always signifies the whole of an ellipse, never a segment of it (*Quaestiones Archimedeae*, p. 149).

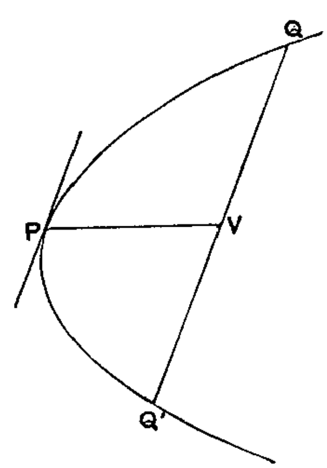
of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that aforesaid. And, in the result, each of the aforesaid theorems has been accepted* no less than those proved without the lemma. As therefore my work now published has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the elementary propositions in conics which are of service in the proof (στοιχεῖα κωνικὰ χρεῖαν ἔχοντα ἐς τὰν ἀπόδειξιν). Farewell."

Proposition 1.

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, *and if* QQ' *be a chord parallel to the tangent to the parabola at* P *and meeting* PV *in* V, *then*

$$QV = VQ'.$$

Conversely, *if* $QV = VQ'$, *the chord* QQ' *will be parallel to the tangent at* P.

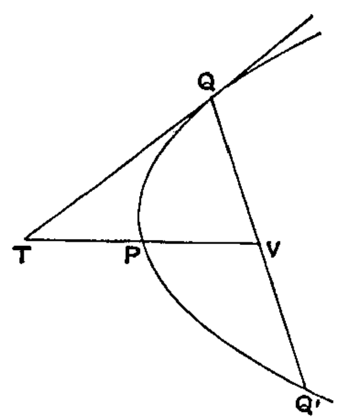


* The Greek of this passage is: συμβαίνει δὲ τῶν προειρημένων θεωρημάτων ἕκαστον μηδὲν ἧσσον τῶν ἄνευ τούτου τοῦ λήμματος ἀποδεδειγμένων πεπιστευκέναι. Here it would seem that πεπιστευκέναι must be wrong and that the passive should have been used.

Proposition 2.

If in a parabola QQ' be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets QQ' in V and the tangent at Q to the parabola in T, then

$$PV = PT.$$



Proposition 3.

If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V' respectively, then

$$PV : PV' = QV^2 : Q'V'^2.$$

"*And these propositions are proved in the elements of conics.**"

Proposition 4.

If Qq be the base of any segment of a parabola, and P the vertex of the segment, and if the diameter through any other point R meet Qq in O and QP (produced if necessary) in F, then

$$QV : VO = OF : FR.$$

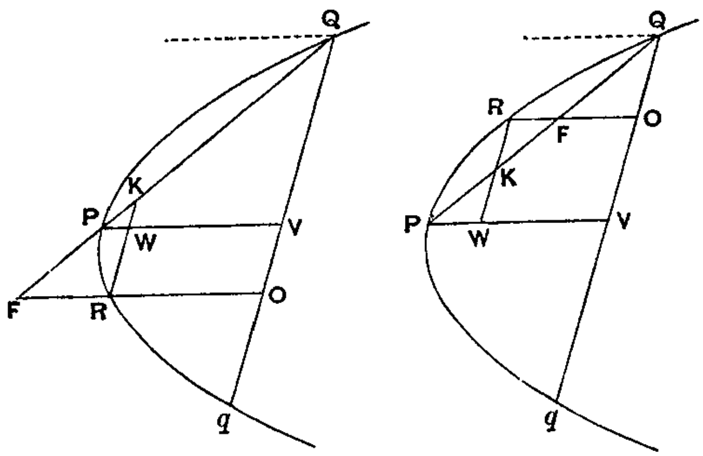
Draw the ordinate RW to PV, meeting QP in K.

* i.e. in the treatises on conics by Euclid and Aristaeus.

Then $$PV : PW = QV^2 : RW^2;$$

whence, by parallels,

$$PQ : PK = PQ^2 : PF^2.$$



In other words, PQ, PF, PK are in continued proportion; therefore

$$PQ : PF = PF : PK$$
$$= PQ \pm PF : PF \pm PK$$
$$= QF : KF.$$

Hence, by parallels,

$$QV : VO = OF : FR.$$

[It is easily seen that this equation is equivalent to a change of axes of coordinates from the tangent and diameter to new axes consisting of the chord Qq (as axis of x, say) and the diameter through Q (as axis of y).

For, if $QV = a$, $PV = \frac{a^2}{p}$, where p is the parameter of the ordinates to PV.

Thus, if $QO = x$, and $RO = y$, the above result gives

$$\frac{a}{x-a} = \frac{OF}{OF - y},$$

whence $$\frac{a}{2a - x} = \frac{OF}{y} = \frac{x \cdot \frac{a}{p}}{y},$$

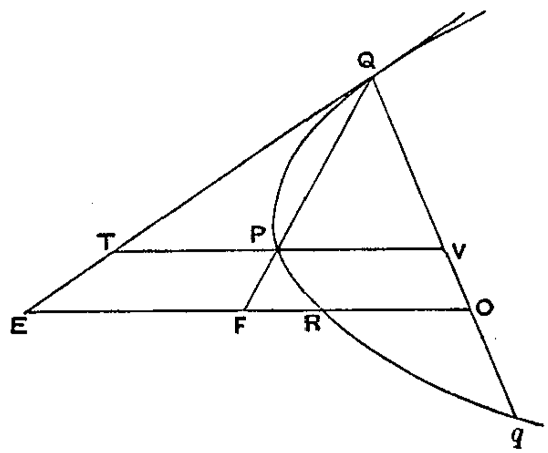
or $$py = x(2a - x).]$$

Proposition 5.

If Qq be the base of any segment of a parabola, P the vertex of the segment, and PV its diameter, and if the diameter of the parabola through any other point R meet Qq in O and the tangent at Q in E, then

$$QO : Oq = ER : RO.$$

Let the diameter through R meet QP in F.



Then, by Prop. 4,

$$QV : VO = OF : FR.$$

Since $QV = Vq$, it follows that

$$QV : qO = OF : OR \qquad \text{..................(1)}.$$

Also, if VP meet the tangent in T,

$$PT = PV, \text{ and therefore } EF = OF.$$

Accordingly, doubling the antecedents in (1), we have

$$Qq : qO = OE : OR,$$

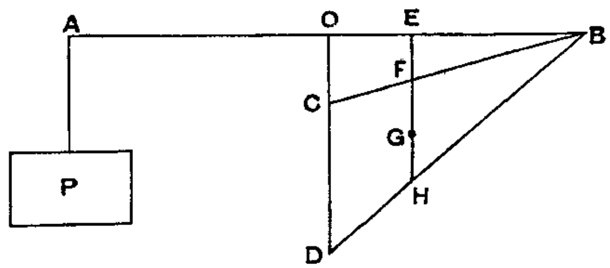
whence $$QO : Oq = ER : RO.$$

Propositions 6, 7*.

Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD in which the angle C is right or obtuse be suspended from B and O, so that C is attached to O and CD is in the same vertical line with O. Then, if P be such an area as, when suspended from A, will keep the system in equilibrium,

$$P = \tfrac{1}{3} \triangle BCD.$$

Take a point E on OB such that $BE = 2OE$, and draw EFH parallel to OCD meeting BC, BD in F, H respectively. Let G be the middle point of FH.



Then G is the centre of gravity of the triangle BCD.

Hence, if the angular points B, C be set free and the triangle be suspended by attaching F to E, the triangle will hang in the same position as before, because EFG is a vertical straight line. "For this is proved†."

Therefore, as before, there will be equilibrium.

Thus $$P : \triangle BCD = OE : AO$$

$$= 1 : 3,$$

or $$P = \tfrac{1}{3} \triangle BCD.$$

* In Prop. 6 Archimedes takes the separate case in which the angle BCD of the triangle is a right angle so that C coincides with O in the figure and F with E. He then proves, in Prop. 7, the same property for the triangle in which BCD is an obtuse angle, by treating the triangle as the difference between two right-angled triangles BOD, BOC and using the result of Prop. 6. I have combined the two propositions in one proof, for the sake of brevity. The same remark applies to the propositions following Props. 6, 7.

† Doubtless in the lost book περὶ ζυγῶν. Cf. the Introduction, Chapter II., *ad fin.*

Propositions 8, 9.

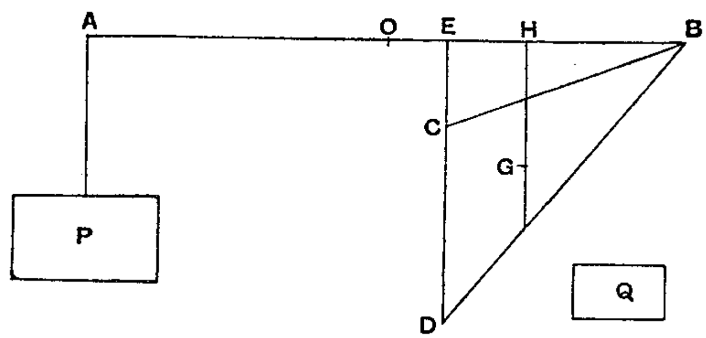
Suppose a lever AOB placed horizontally and supported at its middle point O. Let a triangle BCD, right-angled or obtuse-angled at C, be suspended from the points B, E on OB, the angular point C being so attached to E that the side CD is in the same vertical line with E. Let Q be an area such that

$$AO : OE = \triangle BCD : Q.$$

Then, if an area P suspended from A keep the system in equilibrium,

$$P < \triangle BCD \text{ but } > Q.$$

Take G the centre of gravity of the triangle BCD, and draw GH parallel to DC, i.e. vertically, meeting BO in H.



We may now suppose the triangle BCD suspended from H, and, since there is equilibrium,

$$\triangle BCD : P = AO : OH \quad\text{.................(1)},$$

whence $$P < \triangle BCD.$$

Also $$\triangle BCD : Q = AO : OE.$$

Therefore, by (1), $$\triangle BCD : Q > \triangle BCD : P,$$

and $$P > Q.$$

Propositions 10, 11.

Suppose a lever AOB placed horizontally and supported at O, its middle point. Let CDEF be a trapezium which can be so placed that its parallel sides CD, FE are vertical, while C is vertically below O, and the other sides CF, DE meet in B. Let EF meet BO in H, and let the trapezium be suspended by attaching F to H and C to O. Further, suppose Q to be an area such that

$$AO : OH = (\text{trapezium } CDEF) : Q.$$

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