THE CATTLE-PROBLEM.

IT is required to find the number of bulls and cows of each of four colours, or to find 8 unknown quantities. The first part of the problem connects the unknowns by seven simple equations; and the second part adds two more conditions to which the unknowns must be subject.

Let W, w be the numbers of white bulls and cows respectively,

X, x	,,	,,	black	,,	,,	,,
Y, y	,,	,,	yellow	,,	,,	,,
Z, z	,,	,,	dappled	,,	,,	,,

First part.

(I)

$$W = (\tfrac{1}{2} + \tfrac{1}{3})X + Y \quad\ldots\ldots(\alpha),$$

$$X = (\tfrac{1}{4} + \tfrac{1}{5})Z + Y \quad\ldots\ldots(\beta),$$

$$Z = (\tfrac{1}{6} + \tfrac{1}{7})W + Y \quad\ldots\ldots(\gamma),$$

(II)

$$w = (\tfrac{1}{3} + \tfrac{1}{4})(X + x) \quad\ldots\ldots(\delta),$$

$$x = (\tfrac{1}{4} + \tfrac{1}{5})(Z + z) \quad\ldots\ldots(\epsilon),$$

$$z = (\tfrac{1}{5} + \tfrac{1}{6})(Y + y) \quad\ldots\ldots(\zeta),$$

$$y = (\tfrac{1}{6} + \tfrac{1}{7})(W + w) \quad\ldots\ldots(\eta).$$

Second part.

$$W + X = \text{a square} \quad\ldots\ldots(\theta),$$

$$Y + Z = \text{a triangular number} \quad\ldots\ldots(\iota).$$

[There is an ambiguity in the language which expresses the condition (θ). Literally the lines mean "When the white bulls joined in number with the black, they stood firm (ἔμπεδον) with depth and breadth of equal measurement (ἰσόμετροι εἰς βάθος εἰς εὖρός τε); and the plains of Thrinakia, far-stretching all ways, were filled with their multitude" (reading, with Krumbiegel, πλήθους instead of πλίνθου). Considering that, if the bulls were packed together so as to form a square *figure*, the number of them need not be a square *number*, since a bull is longer than it is broad, it is clear that one possible interpretation would be to take the 'square' to be a square *figure*, and to understand condition (θ) to be simply

$$W+X = \text{a rectangle (i.e. a product of two factors).}$$

The problem may therefore be stated in two forms:

(1) the simpler one in which, for the condition (θ), there is substituted the mere requirement that

$$W+X = \text{a product of two whole numbers;}$$

(2) the complete problem in which all the conditions have to be satisfied including the requirement (θ) that

$$W+X = \text{a square number.}$$

The simpler problem was solved by Jul. Fr. Wurm and may be called

Wurm's Problem.

The solution of this is given (together with a discussion of the complete problem) by Amthor in the *Zeitschrift für Math. u. Physik* (*Hist. litt. Abtheilung*), XXV. (1880), p. 156 sqq.

Multiply (α) by 336, (β) by 280, (γ) by 126, and add; thus

$$297W = 742Y, \text{ or } 3^3 . 11W = 2 . 7 . 53Y \quad\text{.........}(\alpha').$$

Then from (γ) and (β) we obtain

$$891Z = 1580Y, \text{ or } 3^4 . 11Z = 2^2 . 5 . 79Y \quad\text{.......}(\beta'),$$

and

$$99X = 178Y, \text{ or } 3^2 . 11X = 2 . 89Y \quad\text{............}(\gamma').$$

Again, if we multiply (δ) by 4800, (ϵ) by 2800, (ζ) by 1260, (η) by 462, and add, we obtain

$$4657w = 2800X + 1260Z + 462Y + 143W;$$

and, by means of the values in (α'), (β'), (γ'), we derive

$$297 \, . \, 4657w = 2402120Y,$$

or $$3^3 \, . \, 11 \, . \, 4657w = 2^3 \, . \, 5 \, . \, 7 \, . \, 23 \, . \, 373Y \quad \ldots\ldots\ldots(\delta').$$

Hence, by means of (η), (ζ), (ϵ), we have

$$3^2 \, . \, 11 \, . \, 4657y = 13 \, . \, 46489Y \quad \ldots\ldots\ldots(\epsilon'),$$

$$3^3 \, . \, 4657z = 2^2 \, . \, 5 \, . \, 7 \, . \, 761Y \quad \ldots\ldots\ldots(\zeta'),$$

and $$3^2 \, . \, 11 \, . \, 4657x = 2 \, . \, 17 \, . \, 15991Y \quad \ldots\ldots\ldots(\eta').$$

And, since all the unknowns must be whole numbers, we see from the equations (α'), (β'), ... (η') that Y must be divisible by $3^4 \, . \, 11 \, . \, 4657$, i.e. we may put

$$Y = 3^4 \, . \, 11 \, . \, 4657n = 4149387n.$$

Therefore the equations (α'), (β'),...(η') give the following values for all the unknowns in terms of n, viz.

$$\left.\begin{aligned} W &= 2 \, . \, 3 \, . \, 7 \, . \, 53 \, . \, 4657n &&= 10366482n \\ X &= 2 \, . \, 3^2 \, . \, 89 \, . \, 4657n &&= 7460514n \\ Y &= 3^4 \, . \, 11 \, . \, 4657n &&= 4149387n \\ Z &= 2^2 \, . \, 5 \, . \, 79 \, . \, 4657n &&= 7358060n \\ w &= 2^3 \, . \, 3 \, . \, 5 \, . \, 7 \, . \, 23 \, . \, 373n &&= 7206360n \\ x &= 2 \, . \, 3^2 \, . \, 17 \, . \, 15991n &&= 4893246n \\ y &= 3^2 \, . \, 13 \, . \, 46489n &&= 5439213n \\ z &= 2^2 \, . \, 3 \, . \, 5 \, . \, 7 \, . \, 11 \, . \, 761n &&= 3515820n \end{aligned}\right\} \quad \ldots\ldots\ldots(\text{A}).$$

If now $n = 1$, the numbers are the smallest which will satisfy the seven equations (α), (β),...(η); and we have next to find such an integral value for n that the equation (ι) will be satisfied also. [The modified equation (θ) requiring that $W + X$ must be a product of two factors is then simultaneously satisfied.]

Equation (ι) requires that

$$Y + Z = \frac{q(q+1)}{2},$$

where q is some positive integer.

Putting for Y, Z their values as above ascertained, we have

$$\frac{q(q+1)}{2} = (3^4 . 11 + 2^2 . 5 . 79) . 4657n$$
$$= 2471 . 4657n$$
$$= 7 . 353 . 4657n.$$

Now q is either even or odd, so that either $q = 2s$, or $q = 2s - 1$, and the equation becomes

$$s(2s \pm 1) = 7 . 353 . 4657n.$$

As n need not be a prime number, we suppose $n = u . v$, where u is the factor in n which divides s without a remainder and v the factor which divides $2s \pm 1$ without a remainder; we then have the following sixteen alternative pairs of simultaneous equations:

(1)	$s = u$,	$2s \pm 1 = 7 . 353 . 4657v$,
(2)	$s = 7u$,	$2s \pm 1 = 353 . 4657v$,
(3)	$s = 353u$,	$2s \pm 1 = 7 . 4657v$,
(4)	$s = 4657u$,	$2s \pm 1 = 7 . 353v$,
(5)	$s = 7 . 353u$,	$2s \pm 1 = 4657v$,
(6)	$s = 7 . 4657u$,	$2s \pm 1 = 353v$,
(7)	$s = 353 . 4657u$,	$2s \pm 1 = 7v$,
(8)	$s = 7 . 353 . 4657u$,	$2s \pm 1 = v$.

In order to find the least value of n which satisfies all the conditions of the problem, we have to choose from the various positive integral solutions of these pairs of equations that particular one which gives the smallest value for the product uv or n.

If we solve the various pairs and compare the results, we find that it is the pair of equations

$$s = 7u, \qquad 2s - 1 = 353 . 4657v,$$

which leads to the solution we want; this solution is then

$$u = 117423, \qquad v = 1,$$

so that $$n = uv = 117423 = 3^3 . 4349,$$

whence it follows that

$$s = 7u = 821961,$$

and $$q = 2s - 1 = 1643921.$$

Thus
$$\begin{aligned} Y + Z &= 2471 . 4657n \\ &= 2471 . 4657 . 117423 \\ &= 1351238949081 \\ &= \frac{1643921 . 1643922}{2}, \end{aligned}$$

which is a triangular number, as required.

The number in equation (θ) which has to be the product of two integers is now

$$\begin{aligned} W + X &= 2 . 3 . (7 . 53 + 3 . 89) . 4657n \\ &= 2^2 . 3 . 11 . 29 . 4657n \\ &= 2^2 . 3 . 11 . 29 . 4657 . 117423 \\ &= 2^2 . 3^4 . 11 . 29 . 4657 . 4349 \\ &= (2^2 . 3^4 . 4349) . (11 . 29 . 4657) \\ &= 1409076 . 1485583, \end{aligned}$$

which is a rectangular number with nearly equal factors.

The solution is then as follows (substituting for n its value 117423):

$$\begin{aligned} W &= 1217263415886 \\ X &= 876035935422 \\ Y &= 487233469701 \\ Z &= 864005479380 \\ w &= 846192410280 \\ x &= 574579625058 \\ y &= 638688708099 \\ z &= 412838131860 \\ \text{and the sum} &= 5916837175686 \end{aligned}$$

The complete problem.

In this case the seven original equations (α), (β),...(η) have to be satisfied, and the following further conditions must hold,

$$W + X = \text{a square number} = p^2, \text{ say,}$$

$$Y + Z = \text{a triangular number} = \frac{q(q+1)}{2}, \text{ say.}$$

Using the values found above (A), we have in the first place

$$p^2 = 2.3.(7.53 + 3.89).4657n$$

$$= 2^2.3.11.29.4657n,$$

and this equation will be satisfied if

$$n = 3.11.29.4657\xi^2 = 4456749\xi^2,$$

where ξ is any integer.

Thus the first 8 equations (α), (β),...(η), (θ) are satisfied by the following values:

$$W = 2.3^2.7.11.29.53.4657^2.\xi^2 = 46200808287018.\xi^2$$
$$X = 2.3^3.11.29.89.4657^2.\xi^2 = 33249638308986.\xi^2$$
$$Y = 3^5.11^2.29.4657^2.\xi^2 = 18492776362863.\xi^2$$
$$Z = 2^2.3.5.11.29.79.4657^2.\xi^2 = 32793026546940.\xi^2$$
$$w = 2^3.3^2.5.7.11.23.29.373.4657.\xi^2 = 32116937723640.\xi^2$$
$$x = 2.3^3.11.17.29.15991.4657.\xi^2 = 21807969217254.\xi^2$$
$$y = 3^3.11.13.29.46489.4657.\xi^2 = 24241207098537.\xi^2$$
$$z = 2^2.3^2.5.7.11^2.29.761.4657.\xi^2 = 15669127269180.\xi^2$$

It remains to determine ξ so that equation (ι) may be satisfied, i.e. so that

$$Y + Z = \frac{q(q+1)}{2}.$$

Substituting the ascertained values of Y, Z, we have

$$\frac{q(q+1)}{2} = 51285802909803.\xi^2$$

$$= 3.7.11.29.353.4657^2.\xi^2.$$

Multiply by 8, and put

$$2q + 1 = t, \qquad 2 \, . \, 4657 \, . \, \xi = u,$$

and we have the "Pellian" equation

$$t^2 - 1 = 2 \, . \, 3 \, . \, 7 \, . \, 11 \, . \, 29 \, . \, 353 \, . \, u^2,$$

that is, $$t^2 - 4729494 \, u^2 = 1.$$

Of the solutions of this equation the smallest has to be chosen for which u is divisible by $2 \, . \, 4657$.

When this is done,

$$\xi = \frac{u}{2 \, . \, 4657} \text{ and is a whole number;}$$

whence, by substitution of the value of ξ so found in the last system of equations, we should arrive at the solution of the complete problem.

It would require too much space to enter on the solution of the "Pellian" equation

$$t^2 - 4729494 \, u^2 = 1,$$

and the curious reader is referred to Amthor's paper itself. Suffice it to say that he develops $\sqrt{4729494}$ in the form of a continued fraction as far as the period which occurs after 91 convergents, and, after an arduous piece of work, arrives at the conclusion that

$$W = 1598 \, \langle 206541 \rangle,$$

where $\langle 206541 \rangle$ represents the fact that there are 206541 more digits to follow, and that, with the same notation,

$$\text{the whole number of cattle} = 7766 \, \langle 206541 \rangle.$$

One may well be excused for doubting whether Archimedes solved the complete problem, having regard to the enormous

END OF SAMPLE TEXT