BOOK SEVEN

DEFINITIONS

1. An *unit* is that by virtue of which each of the things that exist is called one.

2. A *number* is a multitude composed of units.

3. A number is *a part* of a number, the less of the greater, when it measures the greater;

4. but *parts* when it does not measure it.

5. The greater number is a *multiple* of the less when it is measured by the less.

6. An *even number* is that which is divisible into two equal parts.

7. An *odd number* is that which is not divisible into two equal parts, or that which differs by an unit from an even number.

8. An *even-times even number* is that which is measured by an even number according to an even number.

9. An *even-times odd number* is that which is measured by an even number according to an odd number.

10. An *odd-times odd number* is that which is measured by an odd number according to an odd number.

11. A *prime number* is that which is measured by an unit alone.

12. Numbers *prime to one another* are those which are measured by an unit alone as a common measure.

13. A *composite number* is that which is measured by some number.

14. Numbers *composite to one another* are those which are measured by some number as a common measure.

15. A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

16. And, when two numbers having multiplied one another make some number, the number so produced is called *plane*, and its *sides* are the numbers which have multiplied one another.

17. And, when three numbers having multiplied one another make some number, the number so produced is *solid*, and its *sides* are the numbers which have multiplied one another.

18. A *square number* is equal multiplied by equal, or a number which is contained by two equal numbers.

19. And a *cube* is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

20. Numbers are *proportional* when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

21. *Similar plane* and *solid* numbers are those which have their sides proportional.

22. A *perfect number* is that which is equal to its own parts.

BOOK VII. PROPOSITIONS

Proposition 1

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers AB, CD being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left;

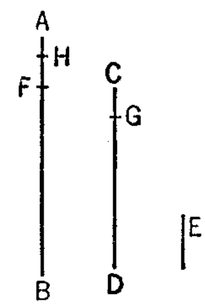
I say that AB, CD are prime to one another, that is, that an unit alone measures AB, CD.

For, if AB, CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E; let CD, measuring BF, leave FA less than itself,

let AF, measuring DG, leave GC less than itself,

and let GC, measuring FH, leave an unit HA.



Since, then, E measures CD, and CD measures BF,

therefore E also measures BF.

But it also measures the whole BA;

therefore it will also measure the remainder AF.

But AF measures DG;

therefore E also measures DG.

But it also measures the whole DC;

therefore it will also measure the remainder CG.

But CG measures FH;

therefore E also measures FH.

But it also measures the whole FA;

therefore it will also measure the remainder, the unit AH, though it is a number: which is impossible.

Therefore no number will measure the numbers AB, CD; therefore AB, CD are prime to one another. [VII. Def. 12]

Q. E. D.

Proposition 2

Given two numbers not prime to one another, to find their greatest common measure.

Let AB, CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB, CD.

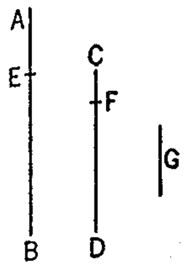
If now CD measures AB—and it also measures itself—CD is a common measure of CD, AB.

And it is manifest that it is also the greatest; for no greater number than CD will measure CD.

But, if CD does not measure AB, then, the less of the numbers AB, CD being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left; otherwise AB, CD will be prime to one another [VII. 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.



Now let CD, measuring BE, leave EA less than itself,
let EA, measuring DF, leave FC less than itself,
and let CF measure AE.

Since then, CF measures AE, and AE measures DF,
therefore CF will also measure DF.

But it also measures itself;
therefore it will also measure the whole CD.

But CD measures BE;
therefore CF also measures BE.

But it also measures EA;
therefore it will also measure the whole BA.

But it also measures CD;
therefore CF measures AB, CD.

Therefore CF is a common measure of AB, CD.

I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB, CD, some number which is greater than CF will measure the numbers AB, CD.

Let such a number measure them, and let it be G.

Now, since G measures CD, while CD measures BE, G also measures BE.

But it also measures the whole BA;
therefore it will also measure the remainder AE.

But AE measures DF;
therefore G will also measure DF.

But it also measures the whole DC;
therefore it will also measure the remainder CF, that is, the greater will measure the less: which is impossible.

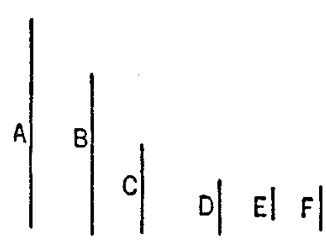
Therefore no number which is greater than CF will measure the numbers AB, CD;
therefore CF is the greatest common measure of AB, CD.

PORISM. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure. Q. E. D.

PROPOSITION 3

Given three numbers not prime to one another, to find their greatest common measure.

Let A, B, C be the three given numbers not prime to one another;
thus it is required to find the greatest common measure of A, B, C.



For let the greatest common measure, D, of the two numbers A, B be taken; [VII. 2]
then D either measures, or does not measure, C.

First, let it measure it.

But it measures A, B also;
therefore D measures A, B, C;
therefore D is a common measure of A, B, C.

I say that it is also the greatest.

For, if D is not the greatest common measure of A, B, C, some number which is greater than D will measure the numbers A, B, C.

Let such a number measure them, and let it be E.

Since then E measures A, B, C,

it will also measure A, B;

therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;

therefore E measures D, the greater the less: which is impossible.

Therefore no number which is greater than D will measure the numbers A, B, C;

therefore D is the greatest common measure of A, B, C.

Next, let D not measure C;

I say first that C, D are not prime to one another.

For, since A, B, C are not prime to one another, some number will measure them.

Now that which measures A, B, C will also measure A, B, and will measure D, the greatest common measure of A, B. [VII. 2, Por.]

But it measures C also;

therefore some number will measure the numbers D, C;

therefore D, C are not prime to one another.

Let then their greatest common measure E be taken. [VII. 2]

Then, since E measures D,

and D measures A, B,

therefore E also measures A, B.

But it measures C also;

therefore E measures A, B, C;

therefore E is a common measure of A, B, C.

I say next that it is also the greatest.

For, if E is not the greatest common measure of A, B, C, some number which is greater than E will measure the numbers A, B, C.

Let such a number measure them, and let it be F.

Now, since F measures A, B, C,

it also measures A, B;

therefore it will also measure the greatest common measure of A, B. [VII. 2, Por.]

But the greatest common measure of A, B is D;

therefore F measures D.

And it measures C also;

therefore F measures D, C;

therefore it will also measure the greatest common measure of D, C. [VII. 2, Por.]

But the greatest common measure of D, C is E;

therefore F measures E, the greater the less: which is impossible

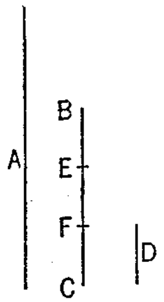
Therefore no number which is greater than E will measure the numbers A, B, C;

therefore E is the greatest common measure of A, B, C.

Q. E. D.

Proposition 4

Any number is either a part or parts of any number, the less of the greater.



Let A, BC be two numbers, and let BC be the less;

I say that BC is either a part, or parts, of A.

For A, BC are either prime to one another or not.

First, let A, BC be prime to one another.

Then, if BC be divided into the units in it, each unit of those in BC will be some part of A; so that BC is parts of A.

Next let A, BC not be prime to one another; then BC either measures, or does not measure, A.

If now BC measures A, BC is a part of A.

But, if not, let the greatest common measure D of A, BC be taken; [VII. 2]
and let BC be divided into the numbers equal to D, namely BE, EF, FC.

Now, since D measures A, D is a part of A.

But D is equal to each of the numbers BE, EF, FC;
therefore each of the numbers BE, EF, FC is also a part of A; so that BC is parts of A.

Therefore etc.

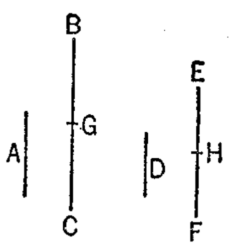
Q. E. D.

Proposition 5

If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

For let the number A be a part of BC,

and another, D, the same part of another EF that A is of BC;



I say that the sum of A, D is also the same part of the sum of BC, EF that A is of BC.

For since, whatever part A is of BC, D is also the same part of EF,
therefore, as many numbers as there are in BC equal to A, so many numbers are there also in EF equal to D.

Let BC be divided into the numbers equal to A, namely BG, GC,

and EF into the numbers equal to D, namely EH, HF;
then the multitude of BG, GC will be equal to the multitude of EH, HF.

And, since BG is equal to A, and EH to D,

therefore BG, EH are also equal to A, D.

For the same reason

GC, HF are also equal to A, D.

Therefore, as many numbers as there are in BC equal to A, so many are there also in BC, EF equal to A, D.

Therefore, whatever multiple BC is of A, the same multiple also is the sum of BC, EF of the sum of A, D.

Therefore, whatever part A is to BC, the same part also is the sum of A, D of the sum of BC, EF.

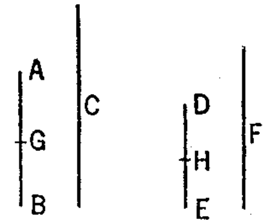
Q. E. D.

Proposition 6

If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

For let the number AB be parts of the number C, and another, DE, the same parts of another, F, that AB is of C;

I say that the sum of AB, DE is also the same parts of the sum of C, F that AB is of C.



For since, whatever parts AB is of C, DE is also the same parts of F,

therefore, as many parts of C as there are in AB, so many parts of F are there also in DE.

Let AB be divided into the parts of C, namely AG, GB, and DE into the parts of F, namely DH, HE;

thus the multitude of AG, GB will be equal to the multitude of DH, HE.

And since, whatever part AG is of C, the same part is DH of F also,

therefore, whatever part AG is of C, the same part also is the sum of AG, DH of the sum of C, F. [VII. 5]

For the same reason,

whatever part GB is of C, the same part also is the sum of GB, HE of the sum of C, F.

Therefore, whatever parts AB is of C, the same parts also is the sum of AB, DE of the sum of C, F. Q. E. D.

Proposition 7

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.

For let the number AB be that part of the number CD which AE subtracted is of CF subtracted;

I say that the remainder EB is also the same part of the remainder FD that the whole AB is of the whole CD.

For, whatever part AE is of CF, the same part also let EB be of CG.

Now since, whatever part AE is of CF, the same part also is EB of CG,

therefore, whatever part AE is of CF, the same part also is AB of GF. [VII. 5]

But, whatever part AE is of CF, the same part also, by hypothesis, is AB of CD;

therefore, whatever part AB is of GF, the same part is it of CD also;

therefore GF is equal to CD.

Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD.

Now since, whatever part AE is of CF, the same part also is EB of GC,

while GC is equal to FD,

therefore, whatever part AE is of CF, the same part also is EB of FD.

But, whatever part AE is of CF, the same part also is AB of CD;

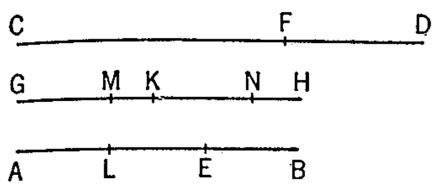
therefore also the remainder EB is the same part of the remainder FD that the whole AB is of the whole CD. Q. E. D.

Proposition 8

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

For let the number AB be the same parts of the number CD that AE subtracted is of CF subtracted;

I say that the remainder EB is also the same parts of the remainder FD that the whole AB is of the whole CD.



For let GH be made equal to AB.

Therefore, whatever parts GH is of CD, the same parts also is AE of CF.

Let GH be divided into the parts of CD, namely GK, KH, and AE into the parts of CF, namely AL, LE;

thus the multitude of GK, KH will be equal to the multitude of AL, LE.

Now since, whatever part GK is of CD, the same part also is AL of CF,

while CD is greater than CF,

therefore GK is also greater than AL.

Let GM be made equal to AL.

Therefore, whatever part GK is of CD, the same part also is GM of CF; therefore also the remainder MK is the same part of the remainder FD that the whole GK is of the whole CD. [VII. 7]

Again, since, whatever part KH is of CD, the same part also is EL of CF,

while CD is greater than CF,

therefore HK is also greater than EL.

Let KN be made equal to EL.

Therefore, whatever part KH is of CD, the same part also is KN of CF; therefore also the remainder NH is the same part of the remainder FD that the whole KH is of the whole CD. [VII. 7]

But the remainder MK was also proved to be the same part of the remainder FD that the whole GK is of the whole CD;

therefore also the sum of MK, NH is the same parts of DF that the whole HG is of the whole CD.

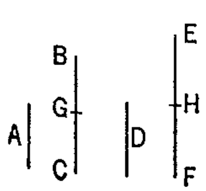
But the sum of MK, NH is equal to EB,

and HG is equal to BA;

therefore the remainder EB is the same parts of the remainder FD that the whole AB is of the whole CD. Q. E. D.

Proposition 9

If a number be a part of a number, and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts, will the second also be of the fourth.



For let the number A be a part of the number BC, and another, D, the same part of another, EF, that A is of BC;

I say that, alternately also, whatever part or parts A is of D, the same part or parts is BC of EF also.

For since, whatever part A is of BC, the same part also is D of EF,