BOOK TEN

DEFINITIONS I

1. Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.

2. Straight lines are *commensurable in square* when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square or in square only, *rational*, but those which are incommensurable with it *irrational*.

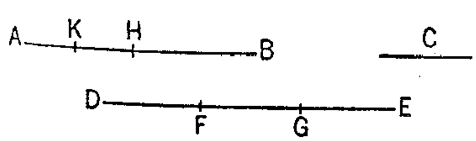
4. And let the square on the assigned straight line be called *rational* and those areas which are commensurable with it *rational*, but those which are incommensurable with it *irrational*, and the straight lines which produce them *irrational*, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

BOOK X. PROPOSITIONS

Proposition 1

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let AB, C be two unequal magnitudes of which AB is the greater:



I say that, if from AB there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the magnitude C.

For C if multiplied will sometime be greater than AB. [cf. v. Def. 4]

Let it be multiplied, and let DE be a multiple of C, and greater than AB;

let DE be divided into the parts DF, FG, GE equal to C,

from AB let there be subtracted BH greater than its half,

and, from AH, HK greater than its half, and let this process be repeated continually until the divisions in AB are equal in multitude with the divisions in DE.

Let, then, AK, KH, HB be divisions which are equal in multitude with DF, FG, GE.

Now, since DE is greater than AB,

and from DE there has been subtracted EG less than its half,

and, from AB, BH greater than its half,

therefore the remainder GD is greater than the remainder HA.

And, since GD is greater than HA,

and there has been subtracted, from GD, the half GF,

and, from HA, HK greater than its half,

therefore the remainder DF is greater than the remainder AK.

But DF is equal to C;

therefore C is also greater than AK.

Therefore AK is less than C.

Therefore there is left of the magnitude AB the magnitude AK which is less than the lesser magnitude set out, namely C. Q. E. D.

And the theorem can be similarly proved even if the parts subtracted be halves.

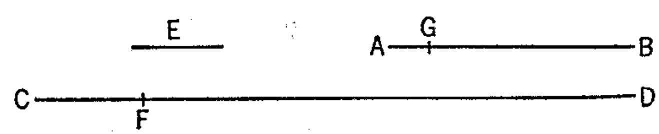
Proposition 2

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

For, there being two unequal magnitudes AB, CD, and AB being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it;

I say that the magnitudes AB, CD are incommensurable.

For, if they are commensurable, some magnitude will measure them.



Let a magnitude measure them, if possible, and let it be E;

let AB, measuring FD, leave CF less than itself,

let CF measuring BG, leave AG less than itself,

and let this process be repeated continually, until there is left some magnitude which is less than E.

Suppose this done, and let there be left AG less than E.

Then, since E measures AB,

while AB measures DF,

therefore E will also measure FD.

But it measures the whole CD also;

therefore it will also measure the remainder CF.

But CF measures BG;

therefore E also measures BG.

But it measures the whole AB also;

therefore it will also measure the remainder AG, the greater the less:

which is impossible.

Therefore no magnitude will measure the magnitudes *AB*, *CD;*
therefore the magnitudes *AB*, *CD* are incommensurable. [x. Def. 1]
Therefore etc. Q. E. D.

Proposition 3

Given two commensurable magnitudes, to find their greatest common measure.

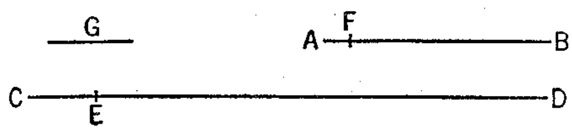
Let the two given commensurable magnitudes be *AB*, *CD* of which *AB* is the less;
thus it is required to find the greatest common measure of *AB*, *CD*.

Now the magnitude *AB* either measures *CD* or it does not.

If then it measures it—and it measures itself also—*AB* is a common measure of *AB*, *CD*.

And it is manifest that it is also the greatest;
for a greater magnitude than the magnitude *AB* will not measure *AB*.

Next, let *AB* not measure *CD*.



Then, if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because *AB*, *CD* are not incommensurable; [cf. x. 2]

let *AB*, measuring *ED*, leave *EC* less than itself,
let *EC*, measuring *FB*, leave *AF* less than itself,
and let *AF* measure *CE*.

Since, then, *AF* measures *CE*,
while *CE* measures *FB*,
therefore *AF* will also measure *FB*.

But it measures itself also;
therefore *AF* will also measure the whole *AB*.

But *AB* measures *DE*;
therefore *AF* will also measure *ED*.

But it measures *CE* also;
therefore it also measures the whole *CD*.

Therefore *AF* is a common measure of *AB*, *CD*.

I say next that it is also the greatest.

For, if not, there will be some magnitude greater than *AF* which will measure *AB*, *CD*.

Let it be *G*.

Since then *G* measures *AB*,
while *AB* measures *ED*,
therefore *G* will also measure *ED*.

But it measures the whole *CD* also;
therefore *G* will also measure the remainder *CE*.

But *CE* measures *FB*;
therefore *G* will also measure *FB*.

But it measures the whole *AB* also,
and it will therefore measure the remainder *AF*, the greater the less:
which is impossible.

Therefore no magnitude greater than AF will measure AB, CD;
therefore AF is the greatest common measure of AB, CD.

Therefore the greatest common measure of the two given commensurable magnitudes AB, CD has been found. Q. E. D.

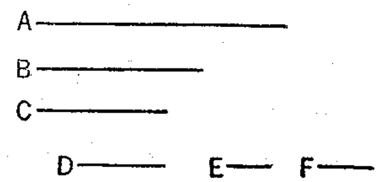
Porism. From this it is manifest that, if a magnitude measure two magnitudes, it will also measure their greatest common measure.

Proposition 4

Given three commensurable magnitudes, to find their greatest common measure.

Let A, B, C be the three given commensurable magnitudes;
thus it is required to find the greatest common measure of A, B, C.

Let the greatest common measure of the two magnitudes A, B be taken, and let it be D; [x. 3]
then D either measures C, or does not measure it.



First, let it measure it.

Since then D measures C,
while it also measures A, B,
therefore D is a common measure of A, B, C.

And it is manifest that it is also the greatest;
for a greater magnitude than the magnitude D does not measure A, B.

Next, let D not measure C.

I say first that C, D are commensurable.

For, since A, B, C are commensurable,
some magnitude will measure them,
and this will of course measure A, B also;
so that it will also measure the greatest common measure of A, B, namely D. [x. 3, Por.]

But it also measures C;
so that the said magnitude will measure C, D;
therefore C, D are commensurable.

Now let their greatest common measure be taken, and let it be E. [x. 3]

Since then E measures D,
while D measures A, B,
therefore E will also measure A, B.

But it measures C also;
therefore E measures A, B, C;
therefore E is a common measure of A, B, C.

I say next that it is also the greatest.

For, if possible, let there be some magnitude F greater than E, and let it measure A, B, C.

Now, since F measures A, B, C,
it will also measure A, B,
and will measure the greatest common measure of A, B. [x. 3, Por.]

But the greatest common measure of A, B is D;
therefore F measures D.

But it measures C also;
therefore F measures C, D;
therefore F will also measure the greatest common measure of C, D. [x. 3, Por.]

But that is E;

therefore F will measure E, the greater the less: which is impossible.

Therefore no magnitude greater than the magnitude E will measure A, B, C; therefore E is the greatest common measure of A, B, C if D do not measure C, and, if it measure it, D is itself the greatest common measure.

Therefore the greatest common measure of the three given commensurable magnitudes has been found.

Porism. From this it is manifest that, if a magnitude measure three magnitudes, it will also measure their greatest common measure.

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended. Q. E. D.

Proposition 5

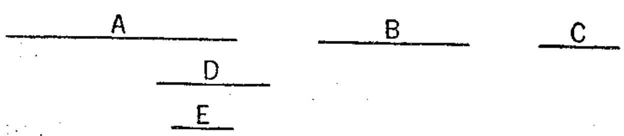
Commensurable magnitudes have to one another the ratio which a number has to a number.

Let A, B be commensurable magnitudes;

I say that A has to B the ratio which a number has to a number.

For, since A, B are commensurable, some magnitude will measure them.

Let it measure them, and let it be C.



And, as many times as C measures A, so many units let there be in D;

and, as many times as C measures B, so many units let there be in E.

Since then C measures A according to the units in D, while the unit also measures D according to the units in it, therefore the unit measures the number D the same number of times as the magnitude C measures A;

therefore as C, is to A, so is the unit to D; [vii. Def. 20]

therefore, inversely, as A is to C, so is D to the unit. [cf. v. 7, Por.]

Again, since C measures B according to the units in E, while the unit also measures E according to the units in it, therefore the unit measures E the same number of times as C measures B;

therefore, as C is to B, so is the unit to E.

But it was also proved that,

as A is to C, so is D to the unit;

therefore, *ex aequali*,

as A is to B, so is the number D to E. [v. 22]

Therefore the commensurable magnitudes A, B have to one another the ratio which the number D has to the number E. Q. E. D.

Proposition 6

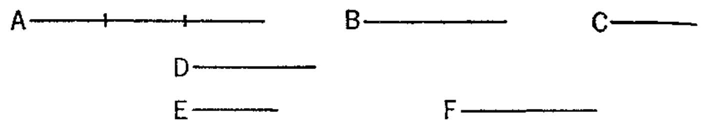
If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

For let the two magnitudes A, B have to one another the ratio which the number D has to the number E;

I say that the magnitudes A, B are commensurable.

For let A be divided into as many equal parts as there are units in D, and let C be equal to one of them;

and let F be made up of as many magnitudes equal to C as there are units in E.



Since then there are in A as many magnitudes equal to C as there are units in D,

whatever part the unit is of D, the same part is C of A also;

therefore, as C is to A, so is the unit to D. [VII. Def. 20]

But the unit measures the number D;

therefore C also measures A.

And since, as C is to A, so is the unit to D,

therefore, inversely, as A is to C, so is the number D to the unit. [cf. V. 7, Por.]

Again, since there are in F as many magnitudes equal to C as there are units in E,

therefore, as C is to F, so is the unit to E. [VII. Def. 20]

But it was also proved that,

as A is to C, so is D to the unit;

therefore, *ex aequali*, as A is to F, so is D to E. [V. 22]

But, as D is to E, so is A to B;

therefore also, as A is to B, so is it to F also. [V. 11]

Therefore A has the same ratio to each of the magnitudes B, F;

therefore B is equal to F. [V. 9]

But C measures F;

therefore it measures B also.

Further it measures A also;

therefore C measures A, B.

Therefore A is commensurable with B.

Therefore etc.

Porism. From this it is manifest that, if there be two numbers, as D, E, and a straight line, as A, it is possible to make a straight line [F] such that the given straight line is to it as the number D is to the number E.

And, if a mean proportional be also taken between A, F, as B,

as A is to F, so will the square on A be to the square on B, that is, as the first is to the third, so is the figure on the first to that which is similar and similarly described on the second. [VI. 19, Por.]

But, as A is to F, so is the number D to the number E;

therefore it has been contrived that, as the number D is to the number E, so also is the figure on the straight line A to the figure on the straight line B.

Q. E. D.

Proposition 7

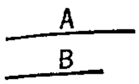
Incommensurable magnitudes have not to one another the ratio which a number has to a number.

Let A, B be incommensurable magnitudes;

I say that A has not to B the ratio which a number has to a number.

For, if A has to B the ratio which a number has to a number, A will be commensurable with B. [x. 6]

But it is not;



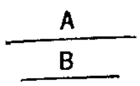
therefore A has not to B the ratio which a number has to a number.

Therefore etc. Q. E. D.

Proposition 8

If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable.

For let the two magnitudes A, B not have to one another the ratio which a number has to a number;



I say that the magnitudes A, B are incommensurable.

For, if they are commensurable, A will have to B the ratio which a number has to a number. [x. 5]

But it has not;

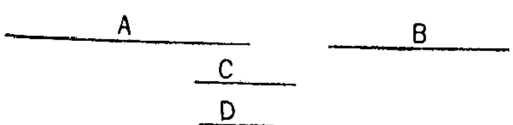
therefore the magnitudes A, B are incommensurable.

Therefore etc. Q. E. D.

Proposition 9

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.

For let A, B be commensurable in length;



I say that the square on A has to the square on B the ratio which a square number has to a square number.

For, since A is commensurable in length with B,

therefore A has to B the ratio which a number has to a number. [x. 5]

Let it have to it the ratio which C has to D.

Since then, as A is to B, so is C to D,

while the ratio of the square on A to the square on B is duplicate of the ratio of A to B,

for similar figures are in the duplicate ratio of their corresponding sides; [vi. 20, Por.]

and the ratio of the square on C to the square on D is duplicate of the ratio of C to D,

for between two square numbers there is one mean proportional number, and the square number has to the square number the ratio duplicate of that which the side has to the side; [viii. 11]

therefore also, as the square on A is to the square on B, so is the square on C to the square on D.

Next, as the square on A is to the square on B, so let the square on C be to the square on D;

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